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Group Inverses of M-Matrices Associated With Nonnegative Matrices Having Few Eigenvalues

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ABSTRACT

We continue here earlier investigations of the structure of group generalized inverses $A^{\#}$ of singular and irreducible M-matrices $A = rI - M$, where M is an $n \times n$ nonnegative and irreducible matrix whose spectral radius is r and where M is assumed to belong to one of several special classes of nonnegative matrices. We now focus on cases where M has only a few distinct eigenvalues. We are particularly interested in instances of such matrices M which cause all or almost all of the off-diagonal entries of the corresponding $A^{\#}$ to be nonpositive. We then apply our results to the study of the sign pattern of the off-diagonal entries of the group generalized inverse of $A = rI - M$, where M comes from one of the following families of nonnegative matrices: (1) the magic square of order $n = 4k$ generated by Matlab, (2) the doubly regular tournament matrices. The latter type is an example of a special type of a nonnegative matrix M for which the group inverse of the associated M-matrix satisfies $A^{\#} = \alpha A^t$ for some $\alpha > 0$.

* Research supported in part by a University of Regina Grad Studies Special Project Grant and NSERC grant OGP0138251.

† Research supported by NSF grants DMS-8901860 and DMS-9007030.

LINEAR ALGEBRA AND ITS APPLICATIONS 220:181–213 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(94)00301-S

1. INTRODUCTION

In this paper we continue our investigation, begun in [17] and [5], of the structure of group generalized inverses of singular and irreducible M-matrices $A = rI - M$, where M is an $n \times n$ nonnegative and irreducible matrix coming from certain special classes of matrices. A result of Deutsch and Neumann [10] asserts that if M is an irreducible nonnegative matrix with rank 1 and spectral radius r, then $(rI - M)^{\#} = (1/r^2)(rI - M)$, so that, in particular, $(rI - M)^{\#}$ is an M-matrix. In the same spirit we will concentrate here on cases where M has few distinct eigenvalues. We are especially interested in the situation where all or most of the off-diagonal entries of the associated group inverse are nonpositive.

We shall apply our results to studying the structure of the group generalized inverse of M-matrices $A = rI - M$, where M comes from one of the following families of matrices:

(1) The magic square of order $4k$ generated by Matlab.
(2) The doubly regular tournament matrices.

The latter type is an example of a special type of nonnegative matrix B for which the group inverse of the associated M-matrix satisfies

$$A^{\#} = \alpha A^{t} \tag{1.1}$$

for some $\alpha > 0$. We call M-matrices which satisfy (1.1) *quasiorthogonal* M-matrices. We characterize the quasiorthogonal M-matrices in terms of their spectra and also explicitly describe the special case of the quasiorthogonal symmetric M-matrices.

We comment that our interest in magic square matrices was kindled by a recent article of Moler [21] in which he posed several problems concerning their eigenvalues and singular values.

2. GROUP INVERSES WHICH ARE M-MATRICES

Throughout this section we shall denote by J_λ the eigenprojection corresponding to the eigenvalue λ of a given matrix.

We begin with the following result:

THEOREM 2.1. *Let M be an $n \times n$ nonnegative and irreducible diagonalizable stochastic matrix. If M has two distinct nonzero eigenvalues* 1 *and* r, *the latter of multiplicity* k, *then*

$$(I-M)^{\#}=I+\frac{k}{k+1-\operatorname{trace}M}M-\frac{2k+1-\operatorname{trace}M}{k+1-\operatorname{trace}M}J_1.$$

In particular, $(I-M)^{\#}$ *is an* M*-matrix if and only if*

$$M_{i,j}\leqslant\frac{(2k+1)-\operatorname{trace}M}{k}(J_1)_{i,j}\qquad\forall i\neq j.\tag{2.1}$$

Proof. From our assumptions it follows that r must be real and that M has the spectral resolution

$$M=J_1+rJ_r.\tag{2.2}$$

Moreover, it is known that

$$I=J_1+J_r+J_0.\tag{2.3}$$

Hence,

$$I-M=(1-r)J_r+J_0.\tag{2.4}$$

But then,

$$\begin{aligned}(I-M)^{\#}&=\frac{1}{1-r}J_r+J_0=I-J_1+\frac{r}{1-r}J_r\\&=I-\frac{1}{r-1}[M-(2-r)J_1]\\&=I+\frac{k}{k+1-\operatorname{trace}M}M-\frac{2k+1-\operatorname{trace}M}{k+1-\operatorname{trace}M}J_1,\end{aligned}\tag{2.5}$$

where the last equality follows from trace $M=kr+1$. Thus the off-diagonal entries of $(I-M)^{\#}$ are nonpositive if and only (2.1) holds. Furthermore, as $(I-M)^{\#}x=0$ for any positive Perron vector of M, we can conclude by [4, Exercise 6.4.14] that $(I-M)^{\#}$ is an M-matrix if and only if (2.1) is true. ∎

Note that Deutsch and Neumann [10] observed that if M is a 2×2 irreducible stochastic matrix, then $(I-M)^{\#}$ an M-matrix. This fact can also be deduced from Theorem 2.1, since if such an M is nonsingular, it is straightforward to verify that (2.1) holds; for in that case, if M is 2×2 and singular, it is necessarily idempotent, so that $(I - M)^{\#} = I - M$, and again $(I - M)^{\#}$ is an M-matrix.

In the special case of M having rank 2, but not necessarily diagonalizable, we still have an analogue to the above theorem.

THEOREM 2.2. *Suppose that M is an $n \times n$ nonnegative irreducible stochastic matrix of rank 2. Then*

$$(I - M)^{\#} = I + \frac{1}{2 - \operatorname{trace} M} M - \frac{3 - \operatorname{trace} M}{2 - \operatorname{trace} M} J_1 .$$

In particular, $(I - M)^{\#}$ is an M-matrix if and only if

$$M_{i,j} \leqslant (3 - \operatorname{trace} M)(J_1)_{i,j} \qquad \forall_i \neq j. \tag{2.6}$$

Proof. There are two cases to consider:

Case 1. M has, apart from the eigenvalue 1, a simple eigenvalue $r \in [-1, 1) \setminus \{0\}$. The result is now (2.1) with $k = 1$.

Case 2. M has a single Jordan block of size 2 corresponding to 0, and all other blocks of M are 1×1. We claim that in this case $(I - M)^{\#} = I + M - 2J_1$. To see the claim, let $Y := I + M - 2J_1$ and note that Y commutes with $I - M$. Further, $(I - M)Y = I - M^2$. It follows from the Jordan form for M that $M^2 = J_1$, so that $(I - M)Y = I - J_1$. Thus $Y = (I - M)^{\#}$. ■

In [5] it was shown that if n is even and M is an $n \times n$ stochastic circulant matrix of the form

$$\frac{2}{n} \operatorname{circ}(\alpha \quad 1 - \alpha \quad \alpha \quad \cdots \quad 1 - \alpha)$$

for some $0 \leqslant \alpha < 1$, then $(I - M)^{\#}$ is an M-matrix if and only if $0 \leqslant \alpha \leqslant \frac{3}{4}$. This fact can now be deduced from Theorem 2.2, since such an M has rank 2 and (2.6) holds if and only if $0 \leqslant \alpha \leqslant \frac{3}{4}$. (Note that J_1 is the matrix whose entries are all $1/n$ in this case.)

In our next theorem we allow for complex eigenvalues.

THEOREM 2.3. *Let M be an $n \times n$ diagonalizable stochastic irreducible nonnegative matrix with one pair of complex eigenvalues λ, $\bar{\lambda}$, each of* *multiplicity* $k = (n-1)/2$. *Then*

$$(I-M)^{\#} = \frac{k+1-\operatorname{trace} M}{k|1-\lambda|^2} I + \frac{1}{|1-\lambda|^2}\left(M - \frac{n-\operatorname{trace} M}{k} J_1\right).$$

In particular, $(I-M)^{\#}$ *is an M-matrix if and only if*

$$M_{i,j} \leqslant \frac{n-\operatorname{trace} M}{k}(J_1)_{i,j} \qquad \forall i \neq j. \tag{2.7}$$

Proof. We write M in terms of its principal idempotents:

$$M = J_1 + \lambda J_\lambda + \bar{\lambda} J_{\bar{\lambda}}.$$

Then $I - M = (1-\lambda)J_\lambda + (1-\bar{\lambda})J_{\bar{\lambda}}$, so that

$$\begin{aligned}
(I-M)^{\#} &= \frac{1}{1-\lambda}J_\lambda + \frac{1}{1-\bar{\lambda}}J_{\bar{\lambda}}\\
&= I - J_1 - J_\lambda - J_{\bar{\lambda}} + \frac{1}{1-\lambda}J_\lambda + \frac{1}{1-\bar{\lambda}}J_{\bar{\lambda}}\\
&= I - J_1 + \frac{\lambda}{1-\lambda}J_\lambda + \frac{\bar{\lambda}}{1-\bar{\lambda}}J_{\bar{\lambda}}\\
&= I - J_1 + \frac{1}{(1-\lambda)(1-\bar{\lambda})}\left[\lambda J_\lambda + \bar{\lambda}J_{\bar{\lambda}} - \lambda\bar{\lambda}(J_\lambda + J_{\bar{\lambda}})\right]\\
&= I - J_1 + \frac{1}{(1-\lambda)(1-\bar{\lambda})}\left[M - J_1 - \lambda\bar{\lambda}(I - J_1)\right]\\
&= \frac{1-2\Re(\lambda)}{|1-\lambda|^2} I + \frac{1}{|1-\lambda|^2}\{M - [2-2\Re(\lambda)]J_1\}.
\end{aligned}$$

Since

$$2\Re(\lambda) = \frac{\text{trace } M - 1}{k},$$

we have that

$$2 - 2\Re(\lambda) = \frac{n - \text{trace } M}{k},$$

and the result follows. ∎

Our next theorem concerns the case where M is nonsingular with three distinct eigenvalues.

THEOREM 2.4. *Let M be an $n \times n$ diagonalizable nonsingular irreducible nonnegative matrix with three distinct eigenvalues: r (the Perron root), s, and t. Then*

$$(rI - M)^{\#} = \frac{r - (s + t)}{(r - s)(r - t)} I + \frac{1}{(r - s)(r - t)} M - \frac{2r - (s + t)}{(r - s)(r - t)} J_r. \tag{2.8}$$

In particular, $(I - M)^{\#}$ is an M-matrix if and only if

$$M_{i,j} \leqslant (2r - s - t)(J_r)_{i,j} \qquad \forall i \neq j. \tag{2.9}$$

Proof. Since $M = rJ_r + sJ_s + tJ_t$ and $I = J_r + J_s + J_t$, it follows that

$$(rI - M)^{\#} = \frac{1}{r - s} J_s + \frac{1}{r - t} J_t. \tag{2.10}$$

We shall next determine scalars α and β such that $\alpha I + \beta M - (\alpha + \beta r)J_r$ is equal to the right-hand side of (2.10). Substituting $rJ_r + sJ_s + tJ_t$ for M and equating the coefficients of J_s and J_t yields the linear system

$$\alpha + s\beta = \frac{1}{r - s},$$

$$\alpha + t\beta = \frac{1}{r - t},$$

which has the solution

$$\alpha = \frac{r-(s+t)}{(r-s)(r-t)} \quad \text{and} \quad \beta = \frac{1}{(r-s)(r-t)}.$$

The expression for $(rI - M)^{\#}$ given in (2.8) now follows readily. We further now see that the off-diagonal entries of $(rI - M)^{\#}$ are nonpositive if the condition (2.9) holds. Finally, if (2.9) holds, the fact that $(rI - M)^{\#}$ is an M-matrix is now a consequence of the fact that for any positive Perron vector x of M, $(rI - M)^{\#}x = 0$. ■

Next we determine a formula for the group inverse of a matrix with a particular eigenstructure. We will apply this result to a certain kind of magic square later on in this paper.

LEMMA 2.5. *Let M be an $n \times n$ diagonalizable matrix of rank* 3 *with nonzero eigenvalues* 1 *and* $\pm x$, $x \neq 1$. *Then*

$$(I-M)^{\#} = I - \frac{3-x^2}{1-x^2}J_1 + \frac{1}{1-x^2}M + \frac{1}{1-x^2}M^2. \tag{2.11}$$

Proof. This time M has the spectral resolution

$$M = J_1 + xJ_x - xJ_{-x}.$$

Furthermore

$$I = J_1 + J_x + J_{-x} + J_0.$$

Hence

$$\begin{aligned}(I-M)^{\#} &= \frac{1}{1-x}J_x + \frac{1}{1+x}J_{-x} + J_0\\ &= \frac{1}{1-x}J_x + \frac{1}{1+x}J_{-x} + I - J_1 - J_x - J_{-x}\\ &= I - J_1 + \frac{1}{1-x^2}(xJ_x - xJ_{-x}) + \frac{1}{1-x^2}(x^2J_x^2 + x^2J_{-x})\\ &= I - J_1 + \frac{1}{1-x^2}(M - J_1) + \frac{1}{1-x^2}(M^2 - J_1).\end{aligned}$$

This last expression can be shown to yield (2.11). ■

We conclude this section with an example illustrating the use of Theorem 2.4. Consider the $n \times n$ matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}. \tag{2.12}$$

According to Aergerter [1] (see also Gregory and Karney [15, p. 74]) the eigenvalues of M are $s := 1$ with multiplicity $n - 2$ and

$$r, t = \frac{1}{2}\left[(n+1) \pm \sqrt{(n+1)^2 + \frac{4n(n+1)(2n-5)}{6}}\right], \tag{2.13}$$

r being the larger of the two and hence the Perron root of M. Our result is:

THEOREM 2.6. *Let M be as in* (2.12). *Then $(I - M)^{\#}$ is an M-matrix.*

Proof. It is readily checked that a Perron vector of M is given by

$$v = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n-1 \\ r-1 \end{pmatrix},$$

so that

$$\begin{aligned} J_r &= \frac{vv^t}{v^tv} \\ &= \frac{1}{v^tv}\begin{bmatrix} 1 & 2 & \cdots & n-1 & r-1 \\ 2 & 4 & \cdots & 2(n-1) & 2(r-1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ (n-1) & 2(n-1) & \cdots & (n-1)^2 & (n-1)(r-1) \\ (r-1) & 2(r-1) & \cdots & (n-1)(r-1) & (r-1)^2 \end{bmatrix}. \end{aligned} \tag{2.14}$$

Since

$$2r - s - t = \frac{n-1}{2} + \frac{3}{2}\sqrt{(n+1)^2 + \frac{4n(n+1)(2n-5)}{6}} \geqslant 0, \tag{2.15}$$

we see that it suffices to verify (2.9) for the last row and column of M. Thus we see that $(rI - M)^{\#}$ is an M-matrix if and only if

$$(2r - s - t)(r - 1) \geqslant v^t v.$$

But this inequality is the same as:

$$\begin{aligned}
&\left(\frac{n-1}{2} + \frac{3}{2}\sqrt{(n+1)^2 + \frac{4n(n+1)(2n-5)}{6}}\right)\\
&\quad\times\left(\frac{n-1}{2} + \frac{1}{2}\sqrt{(n+1)^2 + \frac{4n(n+1)(2n-5)}{6}}\right)\\
&\quad\geqslant \frac{(n-1)n(2n-1)}{6}\\
&\qquad + \left(\frac{n-1}{2} + \frac{1}{2}\sqrt{(n+1)^2 + \frac{4n(n+1)(2n-5)}{6}}\right)^2,
\end{aligned}$$

which is easily verified. Hence $(rI - M)^{\#}$ is an M-matrix. ■

3. APPLICATIONS TO DOUBLY REGULAR TOURNAMENTS

In Theorem 2.3 we derived conditions for the group inverse of $I - M$ to be an M-matrix when M has, apart from the eigenvalue 1, a pair of complex eigenvalues each of multiplicity $k = (n-1)/2$. Note that this necessitates n being odd. An example for such a matrix M which arises in a combinatorial context is derived from a doubly regular tournament matrix.

DEFINITION 3.1 [9]. An $n \times n$ (0, 1) matrix T satisfying

$$T + T^t = J - I,$$

where $J = ee^t$, and

$$T^tT = TT^t = \frac{n+1}{4}I + \frac{n-3}{4}J$$

is called a *doubly regular tournament matrix* or, sometimes, a *Hadamard tournament matrix.*

Doubly regular tournament matrices have the property that each has all its row sums equal to $(n-1)/2$, so that the Perron value of each is $(n-1)/2$, the maximal size of the Perron value among all tournament matrices of this dimension (see [9], for example.) In addition to having the maximum spectral radius, doubly regular tournament matrices also have the property that for any two rows, there are exactly $(n-3)/4$ positions in which both rows have a 1. Note that necessarily $n \equiv 3 \pmod 4$ if there exists a doubly regular tournament matrix of order n. The question of the existence of doubly regular tournament of each permissible order remains open. A particular example, when $n = 7$, is the circulant matrix $T = \text{circ}(0\ 1\ 1\ 0\ 1\ 0\ 0)$.

In a recent paper DeCaen, Gregory, Kirkland, Maybee, and Pullman [9] discussed the eigenvalues of doubly regular tournament matrices T and showed that they are precisely the tournament matrices which possess three distinct eigenvalues: $r = (n-1)/2$, the Perron root of T, and $-\frac{1}{2} \pm i\sqrt{n}/2$. Letting $M = [2/(n-1)]T$, we see from these observations that Theorem 2.3 can be used to examine whether the matrix $I - M$ has the property that $(I - M)^{\#}$ is an M-matrix. When this turns out to be the case, M has the interesting property that $(I - M)^{\#} = \alpha(I - M^t)$ for some $\alpha > 0$. This motivates our next definition.

DEFINITION 3.2. An $n \times n$ stochastic matrix B is called *quasiorthogonal* if

$$(I - B)^{\#} = \alpha(I - B^t)$$

for some $\alpha \in \mathbf{R}$.

We next characterize quasiorthogonal matrices in terms of their eigenvalues.

THEOREM 3.3. *An $n \times n$ stochastic matrix B is quasiorthogonal if and only if B is normal and there exists a constant $\beta > 0$ such that for any eigenvalue $\lambda \neq 1$ of B,*

$$|1 - \lambda|^{-1} = \beta. \tag{3.1}$$

Proof. Suppose first that B is quasiorthogonal. Since $I - B$ commutes with $(I - B)^{\#}$, it follows that B commutes with B^t, so that B is normal. Let $\lambda \neq 1$ be an eigenvalue of B. If $v \neq 0$ is a corresponding eigenvector of B, then it is known that

$$\frac{1}{1-\lambda} v = (I - B)^{\#} v = \alpha (I - B^t) v.$$

Hence

$$\frac{1}{1-\lambda} v^* v = \alpha (1 - \bar{\lambda}) v^* v,$$

so that $|1 - \lambda|^2 = \alpha > 0$. Setting $\beta = \alpha^{-1/2}$ completes the proof of the "only if" part.

Conversely, suppose that B is normal and the eigenvalues of B satisfy (3.1). Let $v_1, \dots, v_n$ be an orthonormal basis of eigenvectors for $\mathbb{C}^n$, with $v_1, \dots, v_k$ eigenvectors corresponding to 1 and with $v_{k+1}, \dots, v_n$ eigenvectors corresponding to $\lambda_{k+1}, \dots, \lambda_n$. Let P be the eigenprojection onto the eigenspace of B corresponding to 1. Then for each $i = 1, \dots, k$,

$$\frac{1}{\beta^2}(I - B)(I - B^t) v_i = 0 = (I - P) v_i, \qquad i = 1, \dots, k.$$

Further, for $i = k+1, \dots, n$,

$$\frac{1}{\beta^2}(I - B)(I - B^t) v_i = \frac{|1 - \lambda_i|^2}{\beta^2} v_i = v_i = (I - P) v_i.$$

Hence

$$\frac{1}{\beta^2}(I - B)(I - B^t) = I - P.$$

It now follows from $(I - B)(I - B)^{\#} = I - P$ that

$$(I - B)^{\#} = \frac{1}{\beta^2}(I - B^t).$$

Letting $\alpha = 1/\beta^2$, the conclusion follows. ■

Our definition of quasiorthogonality was motivated by properties of doubly regular tournament matrices. We will call M a *normalized tournament matrix (derived from T)* if $M = (1/r)T$ for some tournament matrix T with spectral radius r. We next show that the only quasiorthogonal normalized tournament matrices are those derived from doubly regular tournament matrices.

THEOREM 3.4. *A normalized tournament matrix is quasiorthogonal if and only if it is derived from a doubly regular tournament matrix.*

Proof. We need only establish the "only if" part of the claim. Suppose that M is a normalized quasiorthogonal tournament matrix. From the normality of M it follows that M commutes with J and hence the row sums of M are all 1. Since trace $M = 0$ and the real part of any eigenvalue of M other than 1 is at least $-\frac{1}{2}/[(n-1)/2] = -1/(n-1)$ (see [9]), we see that every eigenvalue of M other than 1 has real part equal to $-1/(n-1)$. By Theorem 3.1, all such eigenvalues must lie on a common circle centered at $(1, 0)$, so we see that M must have just two distinct eigenvalues other than 1. As the only tournament matrices with three distinct eigenvalues are doubly regular tournament matrices (see [9]), M must be derived from a doubly regular tournament matrix. ■

Next we characterize nonnegative irreducible stochastic *symmetric* quasiorthogonal matrices.

THEOREM 3.5. *Suppose M is a symmetric stochastic irreducible nonnegative matrix. Then M is quasiorthogonal if and only if*

$$M = rI + (1 - r)\tilde{J}$$

for some r satisfying

$$-\frac{1}{n-1} < r < 1,$$

where $\tilde{J}$ is the $n \times n$ matrix whose entries are all $1/n$.

Proof. The "if" part is obvious, so we only prove the "only if" part. Since M is symmetric, its eigenvalues are real, say 1 and $\lambda_1, \dots, \lambda_{n-1}$. Since M is quasiorthogonal, $(1-\lambda_j)^2 = c$ for some $c > 0$ and $j = 1, \dots, n-1$. Hence $\lambda_1 = \cdots = \lambda_{n-1} := \lambda$, as no λ_i can exceed 1. Resolving $M = J_1 + \lambda J_\lambda$ and taking into account that $J_1 + J_\lambda = I$, we have that

$$M = \lambda I + (1-\lambda) J_1.$$

Finally, since M is symmetric, we have that $J_1 = \tilde{J}$, which yields the result. ■

The final result of this section establishes a connection between certain types of quasiorthogonal matrices and generalized tournament matrices, that is, nonnegative matrices A such that $A + A^t = J - I$.

THEOREM 3.6. *Suppose that M is an $n \times n$ nonnegative stochastic irreducible and normal matrix. Assume further that M has only three distinct eigenvalues*: 1 *and $x \pm iy$, $y \neq 0$. Then the matrix*

$$\frac{n}{2(1-x)}\left[M - \frac{(n-1)r+1}{n} I\right],$$

where r is the non-Perron root of the matrix $(M + M^t)/2$, is a generalized tournament matrix, and, in particular, the diagonal entries of M are all equal to $[(n-1)r+1]/n$.

Proof. For any $0 \leqslant t \leqslant 1$, the matrix $tM + (1-t)M^t$ is normal, irreducible, and stochastic and has just three eigenvalues: 1 and $x \pm i(2t-1)y$ [this is because $Mv = (x+iy)v$ if and only if $M^t v = (x-iy)v$]. Thus $tM + (1-t)M^t$ is quasiorthogonal for any such t. In particular, $(M+M^t)/2$ is quasiorthogonal and symmetric. But then according to the previous theorem,

$$\tfrac{1}{2}(M + M^t) = rI + (1-r)\tilde{J},$$

where $r = x$ is the non-Perron root of $(M+M^t)/2$. Now, $\tilde{J} = (1/n)J$, so that

$$\frac{1}{2}\left[M - \frac{(n-1)r+1}{n} I\right] + \frac{1}{2}\left[M^t - \frac{(n-1)r+1}{n} I\right] = \frac{1-r}{n}(J - I).$$

Letting

$$A := \frac{n}{2(1-r)}\left[M - \frac{(n-1)r+1}{n}I\right],$$

we see that $A + A^t = J - I$. In particular, the diagonal entries of A must be 0, as the diagonal entries of $J - I$ are zero. Furthermore, since the off-diagonal entries of A are nonnegative, it follows that A is a generalized tournament matrix. It now also follows that each diagonal entry of M must equal $[(n-1)r+1]/n$. ∎

Comments.

(i) It can be shown that the only irreducible stochastic quasiorthogonal matrices which are not primitive are those permutionally equivalent to either

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(ii) Certainly, if T is a doubly regular tournament matrix, then any convex combination of T and $\frac{1}{2}(J-I)$ yields a generalized tournament matrix which, when normalized, yields a quasiorthogonal matrix. However, the following example shows that not every generalized tournament matrix which can be normalized to give a quasiorthogonal matrix is of this form: Let T be a doubly regular tournament matrix and

$$M = \begin{pmatrix} T & \frac{1}{2}J \\ \frac{1}{2}J & T \end{pmatrix}.$$

Since T, T^t, and J all commute, it follows that M is normal. Further, for any eigenvector of T corresponding to $\lambda = -\frac{1}{2} \pm i\sqrt{n}/2$, the vectors $w = (v^t \ 0^t)^t$ and $v = (0^t \ v^t)^t$ are both eigenvectors of M corresponding to λ. Thus we see that M has just three distinct eigenvalues—the Perron value and a single complex conjugate pair. Hence by Theorem 3.1, when M is normalized, it yields a quasiorthogonal matrix. But M cannot be written as a convex combination of a doubly regular tournament matrix and $\frac{1}{2}(J-I)$, since M is of order 2 (mod 4), while any doubly regular tournament matrix is of order 3 (mod 4). We pose here the question: *Which generalized tournament matrices can be normalized to yield a quasiorthogonal matrix?*

4. MAGIC SQUARES OF ORDER $n = 4k$

Consider an $n \times n$ matrix S whose entries comprise the integers from 1 to n^2. If those entries are arranged so that the row sums, the column sums, and the sums of the entries on the main diagonal and antidiagonal are all equal, then S is called a *magic square* (of order n).

In [21], C. Moler describes the magic square facility on Matlab and poses several problems. The Matlab magic square case of $n = 4k$ took on interest for us because he indicated that those magic squares have rank 3. We shall prove this by exhibiting a full-rank factorization of the matrix and then use the factorization to obtain an explicit expression for the nonzero eigenvalues. Following this, we shall use Lemma 2.5 to investigate the group inverse of $rI - M$, where M is a magic square of order $n = 4k$ and r is its spectral radius.

The magic square of order $n = 4k$ generated by Matlab can be constructed by the following procedure: Let A be an $n \times n$ matrix whose elements are the integers n^2 down to 1 starting from the top left corner and decreasing as we go down the rows, viz.,

$$A = \begin{pmatrix} n^2 & n^2 - 1 & n^2 - 2 & \cdots & n^2 - n + 1 \\ n^2 - n & n^2 - n - 1 & n^2 - n - 2 & \cdots & n^2 - 2n + 1 \\ \vdots & \vdots & \vdots & & \vdots \\ n & n - 1 & n - 2 & \cdots & 1 \end{pmatrix}.$$

We construct the matrix $\tilde{A}$ from A as follows: We reverse every second and third column, modulo 4, and likewise every second and third row, modulo 4. That is, if we let

$$E = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & ⋰ & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix},$$

then for each $1 \leqslant j \leqslant n$ such that $j \equiv 2 \pmod 4$ or $j \equiv 3 \pmod 4$, replace the jth column of A, c_j, by Ec_j, and similarly, for each $1 \leqslant j \leqslant n$ such that $j \equiv 2 \pmod 4$ or $j \equiv 3 \pmod 4$, replace the jth row of the resulting matrix, r_j^t, by $r_j^t E$. This yields the matrix $\tilde{A}$. The theory of reversions (see Andrews [2]) ensures that the row sums and column sums of $\tilde{A}$ are all equal to $n(n^2 + 1)/2$. Finally, we construct a magic square S_{4k} by permuting the columns of $\tilde{A}$ in order to obtain the desired sums for the main diagonal and antidiagonal. Let P be the permutation matrix whose jth column is e_j if $j \equiv 0 \pmod 4$ or $j \equiv 1 \pmod 4$ and whose jth column is e_{n+1-j} if $j \equiv 2 \pmod 4$ or $j \equiv 3 \pmod 4$. Then the matrix $S_{4k} = P\tilde{A}P^t$ is a magic square of order $4k$. It now follows that the (i, j)th entry of S_{4k} is given by

$$(S_{4k})_{i,j} = \begin{cases} (i-1)n + j & \text{if } \begin{cases} i \equiv 0, 1 \pmod 4 \\ \quad \text{and } j \equiv 2, 3 \pmod 4 \\ \text{or} \\ i \equiv 2, 3 \pmod 4 \\ \quad \text{and } j \equiv 0, 1 \pmod 4, \end{cases} \\ n^2 - (i-1)n - j + 1 & \text{if } \begin{cases} i \equiv 0, 1 \pmod 4 \\ \quad \text{and } j \equiv 0, 1 \pmod 4 \\ \text{or} \\ i \equiv 2, 3 \pmod 4 \\ \quad \text{and } j \equiv 2, 3 \pmod 4. \end{cases} \end{cases}$$

LEMMA 4.1. *The magic square $M = S_{4k}$ of order $4k$ generated by Matlab admits the factorization*

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix} \begin{pmatrix} B_1 & B_2 & \cdots & B_k \end{pmatrix}, \tag{4.1}$$

where

$$A_i = \begin{bmatrix} n^2 & 2 & 3 \\ n+1 & n^2-n-1 & n^2-n-2 \\ 2n+1 & n^2-2n-1 & n^2-2n-2 \\ n^2-3n & 3n+2 & 3n+3 \end{bmatrix} + 4n(i-1)\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \tag{4.2}$$

for $1 \leqslant i \leqslant k$, *and where*

$$B_j = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{bmatrix} + 4(j-1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \quad (4.3)$$

for $1 \leqslant j \leqslant k$.

Proof. Partition M into blocks of size 4×4. To obtain the factorization in (4.1) consider the (1, 1) block of M, viz.,

$$M_{1,1} = \begin{bmatrix} n^2 & 2 & 3 & n^2-3 \\ n+1 & n^2-n-1 & n^2-n-2 & n+4 \\ 2n+1 & n^2-2n-1 & n^2-2n-2 & 2n+4 \\ n^2-3n & 3n+2 & 3n+3 & n^2-3n-3 \end{bmatrix}. \quad (4.4)$$

Next note that the (i, j)th block of M is given by

$$M_{i,j} = M_{1,1} + 4[(j-1)+n(i-1)]\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix},$$

$$1 \leqslant i, j \leqslant k. \quad (4.5)$$

It is now readily verified that $M_{i,j} = A_i B_j$, for all $1 \leqslant i, j \leqslant k$. ■

The above lemma allows us to obtain the eigenvalues of $M = S_{4k}$ as follows:

LEMMA 4.2. *For A and B given in* (4.1)–(4.3),

$$BA = \frac{n}{4}\begin{bmatrix} n^2+n & n^2-n+4 & n^2-n+6 \\ -\frac{n^3}{3}+3n^2-\frac{11n}{3}+5 & \frac{n^3}{3}+n^2+\frac{17n}{3}-7 & \frac{n^3}{3}+n^2+\frac{23n}{3}-13 \\ \frac{n^3}{3}-2n^2+\frac{8n}{3}-3 & -\frac{n^3}{3}-\frac{14n}{3}+5 & -\frac{n^3}{3}-\frac{20n}{3}+9 \end{bmatrix}. \quad (4.6)$$

Moreover, the nonzero eigenvalues of $M = S_{4k}$ *are*

$$r = \frac{n(n^2+1)}{2}, \qquad x = \frac{n}{2\sqrt{3}}\sqrt{n^3-n}, \quad \textit{and} \quad -x. \quad (4.7)$$

Proof. To find the nonzero eigenvalues of M, we recall that if F and G are an $m \times l$ and an $l \times m$ matrix, respectively then $t^l \det(tI - FG) = t^m \det(tI - GF)$ (see, e.g., Schmid [22]). From this fact, it follows that the nonzero eigenvalues of $M = AB$ are the same as the eigenvalues of BA.

Now, $BA = \sum_{i=1}^{k} B_i A_i$, and it is readily verified that

$$B_i A_i = \begin{bmatrix} 2n^2 - 3n & 3n + 4 & 3n + 6 \\ 3n^2 - 8n + 1 & n^2 + 8n + 5 & n^2 + 8n + 7 \\ -3n^2 + 11n + 1 & n^2 - 11n - 7 & n^2 - 11n - 11 \end{bmatrix}$$

$$+ 4n(i-1)\begin{bmatrix} -2 & 2 & 2 \\ -2 & 2 & 2 \\ 4 & -4 & -4 \end{bmatrix}$$

$$+ 4(i-1)\begin{bmatrix} 0 & 0 & 0 \\ 2n^2 - 6n - 2 & -2n^2 + 6n + 6 & -2n^2 + 6n + 10 \\ -(2n^2 - 6n - 2) & -(-2n^2 + 6n + 6) & -(-2n^2 + 6n + 10) \end{bmatrix}$$

$$- 64n(i-1)^2\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}.$$

Summing the terms over $i = 1, \dots, k$ now yields the expression in (4.6) for BA. Since the column sums of BA are equal to

$$\frac{n(n^2+1)}{2},$$

we find that this is an eigenvalue of BA. Furthermore, since

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} BA \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{n}{4}\begin{bmatrix} 2n^2 + 2 & 0 & 0 \\ -\frac{n^3}{3} + 3n^2 - \frac{11n}{3} + 5 & \frac{2n^3}{3} - 2n^2 + \frac{28n}{3} - 12 & \frac{2n^3}{3} - 2n^2 + \frac{34n}{3} - 18 \\ \frac{n^3}{3} - 2n^2 + \frac{8n}{3} - 3 & -\frac{2n^3}{3} + 2n^2 - \frac{22n}{3} + 8 & -\frac{2n^3}{3} + 2n^2 - \frac{28n}{3} + 12 \end{bmatrix}$$

it now follows by a direct calculation that

$$\pm \frac{n}{2\sqrt{3}}\sqrt{n^3 - n}$$

are the remaining eigenvalues of BA and hence they are the remaining nonzero eigenvalues of AB. This concludes the proof. ■

THEOREM 4.3. *Let $M = S_{4k}$ be a magic square of order $n = 4k$. Then the group inverse of $rI - M$ is given by*

$$(rI - M)^{\#} = \frac{1}{r}I - \frac{3r^2 - x^2}{r(r^2 - x^2)}J_r + \frac{1}{r^2 - x^2}M + \frac{1}{r(r^2 - x^2)}M^2, \tag{4.8}$$

where r and x are given in (4.7) *and where*

$$J_r = \tilde{J} = \frac{1}{n}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}. \tag{4.9}$$

Specifically, on partitioning $(rI - M)^{\#}$ into blocks of size 4×4 and on setting

$$y := \frac{n}{r(r^2 - x^2)} = \frac{24}{n^2(n^2+1)(3n^4 - n^3 + 6n^2 + n + 3)}, \tag{4.10}$$

we have that for $1 \leqslant i \leqslant k$, the (i, i)th block of $(rI - M)^{\#}$ is given by

$$\frac{2}{n(n^2+1)}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$-y\left[\begin{matrix} -\frac{5n^3}{12} + n^2 - \frac{n}{12} + \frac{1}{2} & \frac{n^4}{2} + \frac{n^3}{4} - n^2 + \frac{3n}{4} - \frac{1}{2} \\ \frac{n^4}{2} - \frac{n^3}{4} - \frac{n^2}{2} + \frac{n}{4} & \frac{n^3}{12} + \frac{5n^2}{2} - \frac{19n}{12} + 1 \\ \frac{n^4}{2} - \frac{3n^3}{4} - n^2 + \frac{n}{4} & \frac{7n^3}{12} + 3n^2 - \frac{31n}{12} + 1 \\ \frac{13n^3}{12} + \frac{5n^2}{2} - \frac{n}{12} + \frac{1}{2} & \frac{n^4}{2} - \frac{5n^3}{4} - \frac{5n^2}{2} + \frac{15n}{4} - \frac{1}{2} \end{matrix}\right.$$

$$
\left.\begin{matrix}
\frac{n^4}{2} + \frac{n^3}{4} - 2n^2 + \frac{5n}{4} - 1 & -\frac{5n^3}{12} + 4n^2 - \frac{19n}{12} + 2 \\
\frac{n^3}{12} + \frac{7n^2}{2} - \frac{37n}{12} + \frac{3}{2} & \frac{n^4}{2} - \frac{n^3}{4} - \frac{7n^2}{2} + \frac{57n}{12} - \frac{3}{2} \\
\frac{7n^3}{12} + 4n^2 - \frac{61n}{12} + \frac{3}{2} & \frac{n^4}{2} - \frac{3n^3}{4} - 4n^2 + \frac{93n}{12} - \frac{3}{2} \\
\frac{n^4}{2} - \frac{5n^3}{4} - \frac{7n^2}{2} + \frac{87n}{12} - 1 & \frac{13n^3}{12} + \frac{11n^2}{2} - \frac{127n}{12} + 2
\end{matrix}\right]
$$

$$
-2y(i-1)\left[\begin{matrix}
n^3 + 3n^2 - n + 1 & -n^3 - 3n^2 + 3n - 1 \\
-n^3 - 3n^2 + 3n - 1 & n^3 + 3n^2 - 5n + 1 \\
-n^3 - 3n^2 + 5n - 1 & n^3 + 3n^2 - 7n + 1 \\
n^3 + 3n^2 - 7n + 1 & -n^3 - 3n^2 + 9n - 1
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
-n^3 - 3n^2 + 5n - 1 & n^3 + 3n^2 - 7n + 1 \\
n^3 + 3n^2 - 7n + 1 & -n^3 - 3n^2 + 9n - 1 \\
n^3 + 3n^2 - 9n + 1 & -n^3 - 3n^2 + 11n - 1 \\
-n^3 - 3n^2 + 11n - 1 & n^3 + 3n^2 - 13n + 1
\end{matrix}\right]
$$

$$
+16yn(i-1)^2\begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}, \tag{4.11}
$$

while for $i \neq j$, $1 \leqslant i, j \leqslant k$, the (i, j)th off-diagonal block of $(rI - M)^{\#}$ is given by

$$
-y\left[\begin{matrix}
-\frac{5n^3}{12} + n^2 - \frac{n}{12} + \frac{1}{2} & \frac{n^4}{2} + \frac{n^3}{4} - n^2 + \frac{3n}{4} - \frac{1}{2} \\
\frac{n^4}{2} - \frac{n^3}{4} - \frac{n^2}{2} + \frac{n}{4} & \frac{n^3}{12} + \frac{5n^2}{2} - \frac{19n}{12} + 1 \\
\frac{n^4}{2} - \frac{3n^3}{4} + \frac{n}{4} & \frac{7n^3}{12} + 3n^2 - \frac{31n}{12} + 1 \\
\frac{13n^3}{12} + \frac{5n^2}{2} - \frac{n}{12} + \frac{1}{2} & \frac{n^4}{2} - \frac{5n^3}{4} - \frac{5n^2}{2} + \frac{15n}{4} - \frac{1}{2}
\end{matrix}\right.
$$

$$\left.\begin{matrix}
\frac{n^4}{2}+\frac{n^3}{4}-2n^2+\frac{5n}{4}-1 & -\frac{5n^3}{12}+4n^2-\frac{19n}{12}+2 \\
\frac{n^3}{12}+\frac{7n^2}{2}-\frac{37n}{12}+\frac{3}{2} & \frac{n^4}{2}-\frac{n^3}{4}-\frac{7n^2}{2}+\frac{57n}{12}-\frac{3}{2} \\
\frac{7n^3}{12}+4n^2-\frac{61n}{12}+\frac{3}{2} & \frac{n^4}{2}-\frac{3n^3}{4}-4n^2+\frac{93n}{12}-\frac{3}{2} \\
\frac{n^4}{2}-\frac{5n^3}{4}-\frac{7n^2}{2}+\frac{87n}{12}-1 & \frac{13n^3}{12}+\frac{11n^2}{2}-\frac{127n}{12}+2
\end{matrix}\right]$$

$$-2yn(i-1)\left[\begin{matrix}
n^2+n & -n^2-n+2 \\
-(n^2+n) & -(-n^2-n+2) \\
-(n^2+n) & -(-n^2-n+2) \\
n^2+n & -n^2-n+2
\end{matrix}\right.$$

$$\left.\begin{matrix}
(-n^2-n+4) & (n^2+n-6) \\
-(-n^2-n+4) & -(n^2+n-6) \\
-(-n^2-n+4) & -(n^2+n-6) \\
-n^2-n+4 & n^2+n-6
\end{matrix}\right]$$

$$-2y(j-1)\left[\begin{matrix}
2n^2-n+1 & -2n^2+n-1 \\
-2n^2+3n-1 & 2n^2-3n+1 \\
-2n^2+5n-1 & 2n^2-5n+1 \\
2n^2-7n+1 & -2n^2+7n-1
\end{matrix}\right.$$

$$\left.\begin{matrix}
-2n^2+n-1 & 2n^2-n+1 \\
2n^2-3n+1 & -2n^2+3n-1 \\
2n^2-5n+1 & -2n^2+5n-1 \\
-2n^2+7n-1 & 2n^2-7n+1
\end{matrix}\right]$$

$$+16yn(i-1)(j-1)\begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}. \tag{4.12}$$

Proof. Let $\overline{M} = (1/r)M$, and notice that $\overline{M}$ is diagonalizable, since it has rank 3 and has the eigenvalues 1 and $\pm x/r$, where r and x are given in

(4.7). Applying Lemma 2.5 to $\overline{M}$ yields that

$$(I-\overline{M})^{\#} = I - \frac{3-(x/r)^2}{1-(x/r)^2}J_1 + \frac{1}{1-(x/r)^2}\overline{M} + \frac{1}{1-(x/r)^2}\overline{M}^2.$$

Now using the facts that

$$(rI-M)^{\#} = \frac{1}{r}(I-\overline{M})^{\#}$$

and $\overline{M} = (1/r)M$, it is not difficult to establish (4.8).

In order to obtain an explicit expression for $(rI - M)^{\#}$, we need to compute M^2. Now the (i, j)th block of M^2 is given by $A_i(BA)B_j$, and from Lemma 5.2 we have that

$$BA = \frac{n}{4}\begin{bmatrix} n^2+n & n^2-n+4 & n^2-n+6 \\ -\frac{n^3}{3}+3n^2-\frac{11n}{3}+5 & \frac{n^3}{3}+n^2+\frac{17n}{3}-7 & \frac{n^3}{3}+n^2+\frac{23n}{3}-13 \\ \frac{n^3}{3}-2n^2+\frac{8n}{3}-3 & -\frac{n^3}{3}-\frac{14n}{3}+5 & -\frac{n^3}{3}-\frac{20n}{3}+9 \end{bmatrix}.$$

Hence

$$(BA)B_j$$

$$= \frac{n}{4}\begin{bmatrix} n^2+n & n^2-n+4 & n^2-n+6 & n^2+n-6 \\ -\frac{n^3}{3}+3n^2-\frac{11n}{3}+5 & \frac{n^3}{3}+n^2+\frac{17n}{3}-7 & \frac{n^3}{3}+n^2+\frac{23n}{3}-13 & -\frac{n^3}{3}+3n^2-\frac{29n}{3}+23 \\ \frac{n^3}{3}-2n^2+\frac{8n}{3}-3 & -\frac{n^3}{3}-\frac{14n}{3}+5 & -\frac{n^3}{3}-\frac{20n}{3}+9 & \frac{n^3}{3}-2n^2+\frac{26n}{3}-15 \end{bmatrix}$$

$$+n(j-1)\begin{pmatrix} -2 & 2 & 2 & -2 \\ -2n+6 & 2n-6 & 2n-6 & -2n+6 \\ 2n-4 & -2n+4 & -2n+4 & 2n-4 \end{pmatrix}.$$

This, in turn, yields after some calculations that

$$A_i(BA)B_j = \frac{n}{4}\begin{bmatrix} n^4+\frac{4n^3}{3}+\frac{2n}{3}+1 & n^4-\frac{4n^3}{3}+6n^2-\frac{8n}{3}+1 \\ n^4-\frac{4n^3}{3}+6n^2-\frac{8n}{3}+1 & n^4+\frac{4n^3}{3}-4n^2+\frac{26n}{3}+1 \\ n^4-\frac{4n^3}{3}+8n^2-\frac{14n}{3}+1 & n^4+\frac{4n^3}{3}-6n^2+\frac{44n}{3}+1 \\ n^4+\frac{4n^3}{3}-6n^2+\frac{20n}{3}+1 & n^4-\frac{4n^3}{3}+12n^2-\frac{62n}{3}+1 \end{bmatrix.}$$

$$\begin{matrix} n^4-\frac{4n^3}{3}+8n^2-\frac{14n}{3}+1 & n^4+\frac{4n^3}{3}-6n^2+\frac{20n}{3}+1 \\ n^4+\frac{4n^3}{3}-6n^2+\frac{44n}{3}+1 & n^4-\frac{4n^3}{3}+12n^2-\frac{62n}{3}+1 \\ n^4+\frac{4n^3}{3}-8n^2+\frac{74n}{3}+1 & n^4-\frac{4n^3}{3}+14n^2-\frac{104n}{3}+1 \\ n^4-\frac{4n^3}{3}+14n^2-\frac{104n}{3}+1 & n^4+\frac{4n^3}{3}-12n^2+\frac{146n}{3}+1 \end{matrix}\Bigg]$$

$$+n(j-1)\begin{bmatrix} -2n^2+2n & 2n^2-2n & 2n^2-2n & -2n^2+2n \\ 2n^2-6n & -2n^2+6n & -2n^2+6n & 2n^2-6n \\ 2n^2-10n & -2n^2+10n & -2n^2+10n & 2n^2-10n \\ -2n^2+14n & 2n^2-14n & 2n^2-14n & -2n^2+14n \end{bmatrix}$$

$$+n^2(i-1)\begin{bmatrix} -2n+2 & 2n-6 & 2n-10 & -2n+14 \\ -(-2n+2) & -(2n-6) & -(2n-10) & -(-2n+14) \\ -(-2n+2) & -(2n-6) & -(2n-10) & -(-2n+14) \\ -2n+2 & 2n-6 & 2n-10 & -2n+14 \end{bmatrix}$$

$$+16n^2(i-1)(j-1)\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Next, to compute

$$D = (3r^2 - x^2)J_r - rM - M^2 \tag{4.13}$$

we note that

$$3r^2 - x^2 = \frac{3n^2(n^4 + 2n^2 + 1)}{4} - \frac{n^2(n^3 - n)}{12}$$

$$= \frac{n^2}{12}(9n^4 - n^3 + 18n^2 + n + 9).$$

Thus the (i, j)th block, $D_{i,j}$, of $(3r^2 - x^2)J_r - rM - M^2$ is equal to

$$\frac{n^2}{12}(9n^4 - n^3 + 18n^2 + n + 9)\frac{1}{n}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$- \frac{n(n^2+1)}{2}\left\{\begin{bmatrix} n^2 & 2 & 3 & n^2 - 3 \\ n + 1 & n^2 - n - 1 & n^2 - n - 2 & n + 4 \\ 2n + 1 & n^2 - 2n - 1 & n^2 - 2n - 2 & 2n + 4 \\ n^2 - 3n & 3n + 2 & 3n + 3 & n^2 - 3n - 3 \end{bmatrix}\right.$$

$$\left. + 4[j - 1 + n(i - 1)]\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}\right\}$$

$$- \frac{n}{4}\left[\begin{matrix} n^4 + \frac{4n^3}{3} + \frac{2n}{3} + 1 & n^4 - \frac{4n^3}{3} + 6n^2 - \frac{8n}{3} + 1 \\ n^4 - \frac{4n^3}{3} + 6n^2 - \frac{8n}{3} + 1 & n^4 + \frac{4n^3}{3} - 4n^2 + \frac{26n}{3} + 1 \\ n^4 - \frac{4n^3}{3} + 8n^2 - \frac{14n}{3} + 1 & n^4 + \frac{4n^3}{3} - 6n^2 + \frac{44n}{3} + 1 \\ n^4 + \frac{4n^3}{3} - 6n^2 + \frac{20n}{3} + 1 & n^4 - \frac{4n^3}{3} + 12n^2 - \frac{62n}{3} + 1 \end{matrix}\right.$$

$$\left.\begin{matrix} n^4 - \frac{4n^3}{3} + 8n^2 - \frac{14n}{3} + 1 & n^4 + \frac{4n^3}{3} - 6n^2 + \frac{20n}{3} + 1 \\ n^4 + \frac{4n^3}{3} - 6n^2 + \frac{44n}{3} + 1 & n^4 - \frac{4n^3}{3} + 12n^2 - \frac{62n}{3} + 1 \\ n^4 + \frac{4n^3}{3} - 8n^2 + \frac{74n}{3} + 1 & n^4 - \frac{4n^3}{3} + 14n^2 - \frac{104n}{3} + 1 \\ n^4 - \frac{4n^3}{3} + 14n^2 - \frac{104n}{3} + 1 & n^4 + \frac{4n^3}{3} - 12n^2 + \frac{146n}{3} + 1 \end{matrix}\right]$$

$$-n(j-1)\begin{bmatrix} -2n^2+2n & 2n^2-2n & 2n^2-2n & -2n^2+2n \\ 2n-6n & -2n^2+6n & -2n^2+6n & 2n^2-6n \\ 2n-10n & -2n^2+10n & -2n^2+10n & 2n^2-10n \\ -2n^2+14n & 2n^2-14n & 2n^2-14n & -2n^2+14n \end{bmatrix}$$

$$-n^2(i-1)\begin{bmatrix} -2n+2 & 2n-6 & 2n-10 & -2n+14 \\ -(-2n+2) & -(2n-6) & -(2n-10) & -(-2n+14) \\ -(-2n+2) & -(2n-6) & -(2n-10) & -(-2n+14) \\ -2n+2 & 2n-6 & 2n-10 & -2n+14 \end{bmatrix}$$

$$-16n^2(i-1)(j-1)\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$= \frac{n}{12}\begin{bmatrix} -5n^3+12n^2-n+6 & 6n^4+3n^3-12n^2+9n-6 & 6n^4+3n^3-24n^2+15n-12 & -5n^3+48n^2-19n+24 \\ 6n^4-3n^3-6n^2+3n & n^3+30n^2-19n+12 & n^3+42n^2-37n+18 & 6n^4-3n^3-42n^2+57n-18 \\ 6n^4-9n^3-12n^2+3n & 7n^3+36n^2-31n+12 & 7n^3+48n^2-61n+18 & 6n^4-9n^3-48n^2+93n-18 \\ 13n^3+30n^2-n+6 & 6n^4-15n^3-30n^2+45n-6 & 6n^4-15n^3-42n^2+87n-12 & 13n^3+66n^2-127n+24 \end{bmatrix}$$

$$+2n^2(i-1)$$

$$\times\begin{bmatrix} n^2+n & -n^2-n+2 & -n^2-n+4 & n^2+n-6 \\ -(n^2+n) & -(-n^2-n+2) & -(-n^2-n+4) & -(n^2+n-6) \\ -(n^2+n) & -(-n^2-n+2) & -(-n^2-n+4) & -(n^2+n-6) \\ n^2+n & -n^2-n+2 & -n^2-n+4 & n^2+n-6 \end{bmatrix} \qquad (\Delta)$$

$$+2n(j-1)$$

$$\times\begin{bmatrix} 2n^2-n+1 & -2n^2+n-1 & -2n^2+n-1 & 2n^2-n+1 \\ -2n^2+3n-1 & 2n^2-3n+1 & 2n^2-3n+1 & -2n^2+3n-1 \\ -2n^2+5n-1 & 2n^2-5n+1 & 2n^2-5n+1 & -2n^2+5n-1 \\ 2n^2-7n+1 & -2n^2+7n-1 & -2n^2+7n-1 & 2n^2-7n+1 \end{bmatrix}$$

$$-16n^2(i-1)(j-1)\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Equations (4.11) and (4.12) are now readily established from this last expression. ■

Theorem 4.3 leads us to the following interesting conclusion:

THEOREM 4.4. *Let $M = S_{4k}$ and let $r = n(n^2 + 1)/2$. Then the following holds for the off-diagonal entries of $(rI - M)^{\#}$:*

(i) $((rI - M)^{\#})_{i,j} < 0$ *for* $i \neq j$, $1 \leqslant i, j \leqslant k$, *and* $k = 1, 2$.

(ii) *For* $k \geqslant 3$, $((rI - M)^{\#})_{i,j} > 0$ *whenever*

(a) $i = 1$ *and* $j \equiv 1 \pmod 4$ *and*

$$\frac{n}{12}(-5n^3 + 12n^2 - n + 6) + (j - 1)2n(2n^2 - n + 1) < 0$$

or

(b) $j \equiv 0 \pmod 4$ *and*

$$\frac{n}{12}(-5n^3 + 48n^2 - 19n + 24) + (j - 1)2n(2n^2 - n + 1) < 0,$$

and $((rI - M)^{\#})_{i,j} < 0$ *otherwise.*

Proof. For $k = 1$, 2 or, equivalently, $n = 4$, 8, (i) follows directly from analyzing the signs in (4.11) and (4.12).

For $k \geqslant 3$ or, equivalently, $n \geqslant 12$, we note from (4.13) that a negative entry in the (i, j)th block of $(rI - M)^{\#}$ corresponds to a positive entry in the matrix (Δ) above, while a positive entry in the (i, j) block of $(rI - M)^{\#}$ corresponds to a negative entry in (Δ). Examining (Δ), we see that the positions in the 4×4 block can be partitioned into three sets:

$$\mathscr{A}_1 = \{(1,2),(1,3),(2,1),(2,4),(3,1),(3,4),(4,2),(4,3)\},$$

$$\mathscr{A}_2 = \{(2,2),(2,3),(3,2),(3,3),(4,1),(1,4)\},$$

$$\mathscr{A}_3 = \{(1,1),(1,4)\}.$$

Consider an entry in (Δ) whose position falls in $\mathscr{A}_1$. Treating this entry as a function of i, we see that its derivative with respect to i is at most

$$2n^2(-n^2 - n + 6) + 16n^2(j - 1) \leqslant 2n^2(-n^2 + n - 2),$$

which is negative. Thinking of the same entry as a function of j, its derivative with respect to j is at most

$$2n(-2n^2 + 7n - 1) + 16n^2(i - 1) \leqslant 2n(-n - 1),$$

which is again negative. Thus any entry in (Δ) whose position belongs to $\mathscr{A}_1$ is *decreasing* in both i and j. A similar analysis of (Δ) with respect to entries corresponding to positions in $\mathscr{A}_2 \cup \mathscr{A}_3$ is *increasing* in both i and j.

Now let a_1 be an entry of (Δ) corresponding to a position in $\mathscr{A}_1$. Then from our discussion above we see that

$$\begin{aligned} a_1 \geqslant \frac{n}{12}&(6n^4 - 15n^3 - 42n^2 + 87n - 12) \\ &+ 2n^2\frac{n-4}{4}(-n^2 - n) + 2n\frac{n-4}{4}(-2n^2 + n - 1) \\ &+ 16n^2\left(\frac{n-4}{4}\right)^2 = n\left(\frac{n^3}{4} - 5n^2 + \frac{249}{12}n + 1\right) > 0. \end{aligned}$$

This last expression is positive for $n \geqslant 16$; a straightforward verification shows that $a_1 > 0$ when $n = 12$ as well.

Next let a_2 be an entry in (Δ) corresponding to a position in $\mathscr{A}_2$. Then

$$a_2 \geqslant \frac{n}{12}(n^3 + 30n^2 - 19n + 12) > 0.$$

Furthermore, if $i \geqslant 2$ and a_3 is an entry in (Δ) corresponding to a position in $\mathscr{A}_3$, then

$$\begin{aligned} a_3 &\geqslant \frac{n}{12}(-5n^3 + 12n^2 - n + 6) + 2n^2(n^2 + n - 6) \\ &= \frac{19n^4}{12} + 3n^3 - \frac{145n^2}{12} + \frac{n}{2} > 0. \end{aligned}$$

Thus we see that a negative entry in (Δ) can only occur if $i = 1$ and, in that case, only in a position corresponding to $\mathscr{A}_3$. Specifically, we see that a positive entry in the $(1, j)$th block occurs in the $(1, 1)$th position for any j such that

$$\frac{n}{12}(-5n^3 + 12n^2 - n + 6) + 2n(j - 1)(2n^2 - n + 1) < 0,$$

while a positive entry in the (1, 4)th position of the (1, j)th block occurs for any j such that

$$\frac{n}{12}(-5n^3 + 48n^2 - 19n + 24) + 2n(j-1)(2n^2 - n + 1) < 0.$$

We see then that (ii) follows directly from the above observations. ■

Finally, Moler [21] also posed the problem of what the singular values of $M = S_{4k}$ are. We now turn to this question. Since M has a constant row and column sums, we have that

$$Me = M^t e = \frac{n(n^2+1)}{2}e.$$

Clearly $[n(n^2+1)/2]^2$ is the largest eigenvalue of M^tM, and hence $n(n^2 + 1)/2$ is the largest singular value of M. To find the two other singular values of M, note that since $M^tM = B^tA^tAB$ [where A and B are specified in (4.1)–(4.3)], the matrices M^tM and $BB^tA^tA = BM^tA$ have the same nonzero eigenvalues. Thus it suffices to find the eigenvalues of BM^tA. To this end we shall find the eigenvalues of the matrix CBM^tAC^{-1}, where

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.14}$$

Now by (4.1),

$$CBM^tAC^{-1} = \sum_{1 \leqslant i,j \leqslant k} CB_j(M^t)_{i,j}A_iC^{-1}. \tag{4.15}$$

Furthermore, we see that for $1 \leqslant i, j \leqslant k$,

$$CB_j = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{bmatrix} + 4(j-1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \tag{4.16}$$

so that

$$CB_j(M^t)_{i,j}$$

$$= \left\{ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{bmatrix} + 4(j-1) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \right\}$$

$$\times \left\{ \begin{bmatrix} n^2 & n+1 & 2n+1 & n^2-3n \\ 2 & n^2-n-1 & n^2-2n-1 & 3n+2 \\ 3 & n^2-n-2 & n^2-2n-2 & 3n+3 \\ n^2-3 & n+4 & 2n+4 & n^2-3n-3 \end{bmatrix} \right.$$

$$\left. +4[j-1+n(i-1)] \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 2n^2+2 & 2n^2+2 & 2n^2+2 & 2n^2+2 \\ 3n^2-7 & n^2+2n+11 & n^2+4n+11 & 3n^2-6n-7 \\ -3n^2+12 & n^2-4n-14 & n^2-8n-14 & -3n^2+2n+12 \end{bmatrix}$$

$$+ 4(j-1) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2n^2-8 & -2n^2+4n+8 & -2n^2+8n+8 & 2n^2-12n-8 \\ -(2n^2-8) & -(-2n^2+4n+8) & -(-2n^2+8n+8) & -(2n^2-12n-8) \end{bmatrix}$$

$$+ 4[j-1+n(i-1)] \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 2 & -2 \\ 4 & -4 & -4 & 4 \end{bmatrix}$$

$$+ 16(j-1)[j-1+n(i-1)] \begin{bmatrix} 0 & 0 & 0 & 0 \\ -4 & 4 & 4 & -4 \\ 4 & -4 & -4 & 4 \end{bmatrix}.$$

Consequently,

$$CB_j(M')_{i,j}A_iC^{-1}$$

$$= CB_j(M')_{i,j}\left\{\begin{bmatrix} n^2 & -n^2+2 & -n^2+3 \\ n+1 & n^2-2n-2 & n^2-2n-3 \\ 2n+1 & n^2-4n-2 & n^2-4n-3 \\ n^2-3n & -n^2+6n+2 & -2n^2+6n+3 \end{bmatrix}\right.$$

$$\left. +4n(i-1)\begin{bmatrix} -1 & 2 & 2 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \\ -1 & 2 & 2 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} (2n^2+2)^2 & 0 & 0 \\ * & -4n^4+24n^3-12n^2-132n-72 & -4n^4+24n^3-8n^2-144n-108 \\ * & 8n^4-48n^3+44n^2+204n+104 & 8n^4-48n^3+36n^2+228n+156 \end{bmatrix}$$

$$+ 4n(i-1)\begin{bmatrix} 0 & 0 & 0 \\ * & 8n^2-24n-72 & 8n^2-24n-72 \\ * & -16n^2+48n+104 & -16n^2+48n+104 \end{bmatrix}$$

$$+ 16n(i-1)(j-1)\begin{bmatrix} 0 & 0 & 0 \\ * & 16n^2-48n-64 & 16n^2-48n-64 \\ * & -(16n^2-48n-64) & -(16n^2-48n-64) \end{bmatrix}$$

$$+ 4(j-1)\begin{bmatrix} 0 & 0 & 0 \\ * & -8n^4+48n^3-64n^2-144n-64 & -8n^4+48n^2-56n^2-168n-96 \\ * & -(-8n^4+48n^3-64n^2-144n-64) & -(-8n^4+48n^2-56n^2-168n-96) \end{bmatrix}$$

$$+ 4[j-1+n(i-1)]\begin{bmatrix} 0 & 0 & 0 \\ * & 8n^2-24n-16 & 8n^2-24n-24 \\ * & -16n^2+48n+32 & -16n^2+48n+48 \end{bmatrix}$$

$$+16n(i-1)[j-1+n(i-1)]\begin{bmatrix} 0 & 0 & 0 \\ * & -16 & -16 \\ * & 32 & 32 \end{bmatrix}$$

$$+ 16(j-1)[j-1+n(i-1)]$$

$$\times \begin{bmatrix} 0 & 0 & 0 \\ * & 16n^2 - 48n - 32 & 16n^2 - 48n - 48 \\ * & -(16n^2 - 48n - 32) & -(16n^2 - 48n - 48) \end{bmatrix}$$

$$+64n(i-1)[(j-1)+n(i-1)] \begin{bmatrix} 0 & 0 & 0 \\ * & -32 & -32 \\ * & 32 & 32 \end{bmatrix}.$$

To determine the last two singular values of M we need only look at the bottom right-hand 2×2 submatrix in the above expression. Summing those 2×2 matrices over all $1 \leqslant i, j \leqslant k$ yields the matrix

$$\frac{n^2}{12} \begin{bmatrix} -n^5 + 3n^4 + 2n^3 - 5n^2 - n + 2 & -n^5 + 3n^4 + 2n^3 - 6n^2 - n + 3 \\ n^5 - 2n^4 - 2n^3 + 4n^2 + n - 2 & n^5 - 2n^4 - 2n^3 + 5n^2 + n - 3 \end{bmatrix}. \tag{4.17}$$

Since the column sums of this last matrix are both

$$\frac{n^2(n^4 - n^2)}{12},$$

we see that $n^2(n^4 - n^2)/12$ is one eigenvalue; from the trace, it follows that $n^2(n^2 - 1)/12$ is the other eigenvalue. We have proved:

THEOREM 4.5. *Let $M = S_{4k}$ be the magic square of order $n = 4k$ generated by Matlab. Then the singular values of M are*

$$\frac{n(n^2+1)}{2}, \quad \frac{n}{2}\sqrt{\frac{n^4 - n^2}{3}}, \quad \textit{and} \quad \frac{n}{2}\sqrt{\frac{n^2-1}{3}}. \tag{4.18}$$

The authors would like to acknowledge the careful reading which the referee gave the manuscript and several useful suggestions for its polishing. In particular, he / she improved the utility of the original statement of Theorem 2.4.

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Received 30 July 1993; final manuscript accepted 30 October 1994